FLOW ROUND PERMEABLE CONTOURS

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1. Stating the problem. We consider the problem of the flow round an arbitrary, smooth, closed, uniformly permeable contour in a potential stream of ideal fluid. The problem will be solved under the hypotheses formulated by Rakhmatulin [3]:

1. At all points of the contour, normal components of fluid velocity are continuous; the tangential velocity components, however, are allowed to be discontinuous. Consequently, a permeable contour turns out to be a line of discontinuity in velocity and pressure.

2. The flow is assumed to be steady and irrotational.

3. The contour is permeable uniformly and at every point of the contour the pressure drop Δp and velocity of penetration v_i^0 of fluid particles across the contour are connected by the rule:

$$\Delta p = a v_i^\circ \tag{1.1}$$

where a is a parameter of permeability of the contour material, to be determined experimentally. We now formulate the boundary conditions of the problem.

1. According to the first fundamental hypothesis, at all points of the permeable contour the normal component of velocity v_{n1}^0 of the external flow is equal to the normal component of velocity v_{n2}^0 of the internal flow, i.e.

$$v_{n1}^\circ = v_{n2}^\circ \tag{1.2}$$

The suffices 1 and 2 will be used to indicate quantities in the external and internal fluid domains respectively.

2. From the second fundamental hypothesis it follows that the Bernoulli-Euler equation is satisfied along a streamline of the flow. Accordingly, from this equation and the first boundary condition (1.2), for a pressure drop Δp at a point on the contour, we have

$$\Delta p = \tfrac{1}{2} \rho (v_{\tau 2}^\circ + v_{\tau 1}^\circ)(v_{\tau 2}^\circ - v_{\tau 1}^\circ)$$

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where $v_{\tau 1}$ and $v_{\tau 2}$ are tangential components of velocity at points on the contour. Taking account of (1.1), we get the second boundary condition:

$$\frac{\rho}{2}(v_{\tau 2}{}^{\circ}+v_{\tau 1}{}^{\circ})(v_{\tau 2}{}^{\circ}-v_{\tau 1}{}^{\circ})=av_{i}{}^{\circ} \tag{1.3}$$

To solve the problem we replace the permeable contour by a vortex layer, the density of which we shall choose to fulfill the boundary condition.

Therefore let L (see Figure) be the uniformly permeable smooth contour.

We denote by $\gamma = \gamma(s)$ the density of the vortex layer regarded as a function of the curvilinear coordinate s along the contour L. Let the functions

$$x = x(s), \qquad y = y(s) \tag{1.4}$$

be parametric equations of our contour.

We denote by s_0 and s the curvilinear coordinates, and by t and t_0 the corresponding complex coordinates, of a fixed and general point respectively on our contour. In what follows we shall call the points themselves corresponding coordinates.

For a positive direction of circuit on L we take a direction such that the domain bounded by the contour L lies on the left.

We denote by ν and θ the angles which the tangents to the contour at the points t_0 and t make with the x-axis, and by θ the angle which the chord passing through these points make with the x-axis.

If ds is the length of an element of arc of the contour, then the vortex strength on this element will be $\gamma(s)ds$. Hence the complex velocity at some point z, induced by the vortex layer distributed along the contour, is

$$(v_x{}^{\circ}-iv_y{}^{\circ})_b=\frac{1}{2\pi i}\int_L\frac{\gamma(s)\,ds}{z-t}$$

Taking into consideration that $ds = e^{-i\theta}dt$, we get

$$(v_x{}^{\circ}-iv_y{}^{\circ})_b=-\frac{1}{2\pi i}\int_L\frac{\gamma(s)\,e^{-i\theta}}{t-z}dt \tag{1.5}$$

The integral on the right is a Cauchy type of integral for the function $\gamma(s)e^{-i\theta}$ which Lavrentev [1] has called a "vortex function".

We shall find the limiting values of the complex velocity at the point t_0 of the contour, suitably denoting the limiting values: when approaching from the side of the positive direction of the normal by the index plus, and when approaching from the side of the negative direction of the normal by the index minus.

According to the formula of Sokhotski-Plemel for the limiting value of the Cauchy-type integral, we have

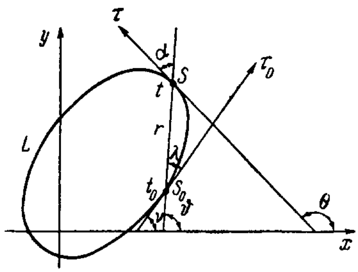

Fig. 1.

$$(v_x^\circ - i v_y^\circ)_{b+} = -\left[\frac{\gamma(s_0)}{2} e^{-i\nu} + \frac{1}{2\pi i}\int\limits_L \frac{\gamma(s)\, e^{-i\theta}}{t - t_0}\, dt\right]$$

$$(v_x^\circ - i v_y^\circ)_{b_-} = -\left[-\frac{\gamma(s_0)}{2} e^{-i\nu} + \frac{1}{2\pi i}\int\limits_L \frac{\gamma(s)\, e^{-i\theta}}{t - t_0}\, dt\right] \tag{1.6}$$

We now change the form of the integral

$$J = \frac{1}{2\pi i}\int\limits_L \frac{\gamma(s)\, e^{-i\theta}}{t - t_0}\, dt \tag{1.7}$$

Taking logarithms and differentiating the identity $t - t_0 = re^{i\theta}$, we get

$$\frac{dt}{t - t_0} = \frac{dr}{r} + i d\vartheta \tag{1.8}$$

We have immediately from the figure

$$\cos\vartheta = \frac{x(s) - x(s_0)}{r}, \qquad \sin\vartheta = \frac{y(s) - y(s_0)}{r}$$

$$\frac{dx(s)}{ds} = \cos\theta, \quad \frac{dy(s)}{ds} = \sin\theta, \quad \frac{dx(s_0)}{ds_0} = \cos\nu, \quad \frac{dy(s_0)}{ds_0} = \sin\nu \tag{1.9}$$

in addition to which

$$\vartheta = \operatorname{arc\,tg} \frac{y(s) - y(s_0)}{x(s) - x(s_0)} \tag{1.10}$$

Hence, taking into consideration the identities (1.9), we get

$$\frac{d\vartheta}{ds} = \frac{\sin\alpha}{r} \qquad (\alpha = \theta - \vartheta) \tag{1.11}$$

Consequently we have

$$dr = \frac{dr}{ds} ds = \cos\alpha\, ds, \qquad d\vartheta = \frac{d\vartheta}{ds} ds = \frac{\sin\alpha}{r} ds$$

Substituting this into the identity (1.8) gives

$$\frac{dt}{t - t_0} = \frac{e^{i\alpha}}{r} ds$$

and the integral (1.7) may be written thus:

$$J = \frac{1}{2\pi i}\int_L \frac{\gamma(s)\, e^{-i\vartheta}}{r}\, ds$$

Substituting this into the equations (1.6) and comparing the real and imaginary parts of these equations, we obtain:

$$\begin{aligned}
(v_x{}^\circ)_{b_+} &= -\frac{\gamma(s_0)}{2}\cos\nu + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\vartheta}{r}\, ds \\
(v_y{}^\circ)_{b_+} &= -\frac{\gamma(s_0)}{2}\sin\nu - \frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\vartheta}{r}\, ds \\
(v_x{}^\circ)_{b_-} &= \frac{\gamma(s_0)}{2}\cos\nu + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\vartheta}{r}\, ds \\
(v_y{}^\circ)_{b_-} &= \frac{\gamma(s_0)}{2}\sin\nu - \frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\vartheta}{r}\, ds
\end{aligned} \tag{1.12}$$

We shall now use the expressions relating the normal and tangential components of velocity at a point on the contour with the components along the axes:

$$v_\tau{}^\circ = v_x{}^\circ \cos\nu + v_y{}^\circ \sin\nu, \qquad v_n{}^\circ = -v_x{}^\circ \sin\nu + v_y{}^\circ \cos\nu \tag{1.13}$$

Then according to (1.12), putting $\lambda = \theta - \nu$, we get

$$(v_\tau{}^\circ)_{b_+} = -\frac{\gamma(s_0)}{2} + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\, ds, \qquad (v_n{}^\circ)_{b_+} = -\frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\lambda}{r}\, ds \tag{1.14}$$

$$(v_\tau{}^\circ)_{b_-} = \frac{\gamma(s_0)}{2} + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\, ds, \qquad (v_n{}^\circ)_{b_-} = -\frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\lambda}{r}\, ds \tag{1.15}$$

The normal v_n and tangential v_τ velocity components of the incident stream are continuous at all points of the contour and consequently, for both the external and internal parts of the stream, are expressed by the formulae:

$$v_\tau = v_\infty \cos\nu, \qquad v_n = -v_\infty \sin\nu \tag{1.16}$$

Comparing the second of the equations (1.14) and (1.15) and remembering that they are components relative only to the incident stream, we see that the first boundary condition (1.2) of the problem is satisfied.

From the first identities of (1.14), (1.15) and (1.16) we get

$$v_{\tau 1}{}^\circ = (v_\tau{}^\circ)_{b_-} + v_\tau = \frac{\gamma(s_0)}{2} + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\, ds + v_\infty\cos\nu$$

$$v_{\tau 2}{}^\circ = (v_\tau{}^\circ)_{b_+} + v_\tau = -\frac{\gamma(s_0)}{2} + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\, ds + v_\infty\cos\nu$$

From the second identities of (1.15) and (1.16) we have

$$v_i{}^\circ = (v_n{}^\circ)_b + v_n = -\frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\lambda}{r}\, ds - v_\infty\sin\nu$$

Substituting the expressions found for the velocity components of the flow under consideration into the second boundary condition (1.3) of our problem, we have

$$\gamma(s_0)\rho\left(v_\infty \cos\nu + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\,ds\right) - \frac{a}{2\pi}\int_L \gamma(s)\frac{\cos\lambda}{r}\,ds = av_\infty \sin\nu \quad (1.17)$$

This is a singular integral equation for the particular determination of the function $\gamma(s_0)$, the density of the vortical layer in our problem.

The parameter of permeability a increases with the compactness of the permeable contour. For a compact contour $a = \infty$. As $a \to \infty$ the equation (1.17) reverts to the equation

$$\frac{1}{2\pi}\int_L \gamma(s)\frac{\cos\lambda}{r}\,ds = -v_\infty \sin\nu \quad (1.18)$$

obtained by Lavrent'ev [1], and solves the problem of a closed compact contour in a potential flow.

2. A method of solving the basic equation (1.17). Differentiating the identity (1.10) with respect to s_0 and using the identity (1.9), we have

$$\frac{d\vartheta}{ds_0} = \frac{\sin\lambda}{r} \quad (2.1)$$

In addition to this

$$\frac{d\ln r}{ds_0} = \frac{1}{r}\frac{dr}{ds_0} = \frac{1}{r}\cos\sigma = -\frac{\cos\lambda}{r}$$

On the other hand

$$r = (t - t_0)\,e^{-i\vartheta}, \qquad \frac{d\ln r}{ds_0} = -\frac{dt_0/ds_0}{t - t_0} - i\frac{d\vartheta}{ds_0} = -\frac{e^{i\nu}}{t - t_0} - i\frac{d\vartheta}{ds_0}$$

Consequently

$$\frac{\cos\lambda}{r} = \frac{e^{i\nu}}{t - t_0} + i\frac{d\vartheta}{ds_0} \quad (2.2)$$

On the basis of the identities (2.1) and (2.2), the equation (1.17) may be written thus:

$$\gamma(s_0)\rho\left(v_\infty\cos\nu + \frac{1}{2\pi}\int_L \gamma(s)\frac{d\vartheta}{ds_0}\,ds\right) - \frac{ae^{i\nu}}{2\pi}\int_L \frac{\gamma(t)\,e^{-i\vartheta}}{t - t_0}\,dt =$$

$$= av_\infty \sin\nu + \frac{ai}{2\pi}\int_L \gamma(s)\frac{d\vartheta}{ds_0}\,ds \quad (2.3)$$

Since ds and ds_0 on one and the same contour L will be equivalent, then

$$\frac{1}{2\pi}\int_L \gamma(s)\frac{d\vartheta}{ds_0}\,ds = \frac{1}{2\pi}\int_L \gamma\,[s(\vartheta)]\,d\vartheta = K = \text{const} \quad (2.4)$$

Furthermore, the circulation Γ of the velocity of the stream around the contour will equal

$$\Gamma = \int_L \left(v_{x1}^\circ \frac{dx}{ds_0} + v_{y1}^\circ \frac{dy}{ds_0} \right) ds_0$$

At the same time it is clear that

$$v_{x1}^\circ = (v_x^\circ)_{b_-} + v_x, \qquad v_{y1}^\circ = (v_y^\circ)_{b_-} + v_y \qquad (v_x = v_\infty, \quad v_y = 0)$$

Then according to the latter pair of relations in (1.12) and the last two identities in (1.9), we have

$$\Gamma = \frac{1}{2} \int_L \gamma(s_0)\, ds_0 + v_\infty \int_L \cos \nu\, ds_0 + \int_L \left[\frac{1}{2\pi} \int_L \gamma(s) \frac{\sin \lambda}{r}\, ds \right] ds_0 \qquad (2.5)$$

On the basis of the identities (2.1), (2.4) and the obvious relations

$$\frac{1}{2} \int_L \gamma(s_0)\, ds_0 = Q = \text{const}, \qquad \int_L \cos \nu\, ds = 0 \qquad (2.6)$$

we get

$$\Gamma = Q + Kl \qquad (2.7)$$

where l is the length of the contour L.

From the expressions (2.4) and (2.6) for the values of K and Q it follows that they are of the same sign and will vanish simultaneously. Thus the constant K is immediately related to the circulation Γ around the contour by the identity (2.7) and becomes zero when $\Gamma = 0$.

Since the value of the circulation Γ for the flow round smooth closed bodies might be given arbitrarily, we may advance arbitrarily a constant quantity K, defining thereupon the appropriate value of the circulation Γ according to the formula (2.7).

Now taking account of (2.4), the equation (2.3) may be written thus:

$$\gamma(t_0)\, \rho\, (v_\infty \cos \nu + K) + \frac{1}{\pi i} \int_L \gamma(t) \frac{(-i) \cdot a e^{-i(\theta - \nu)}}{2(t - t_0)}\, dt = a v_\infty \sin \nu + iaK \qquad (2.8)$$

For this equation we have

$$A(t) = \rho\, (v_\infty \cos \theta + K), \qquad K(t_0, t) = -\frac{a}{2} e^{-i(\theta - \nu)}$$

the functions $A(t)$ and $K(t_0, t)$ satisfying the condition H (Holder) on the contour L

$$B(t) = K(t, t) = -\frac{1}{2} ai$$

$$S(t) = A(t) + B(t) = \rho\, (v_\infty \cos \theta + K) - \frac{1}{2} ai \neq 0 \text{ on } L$$

$$D(t) = A(t) - B(t) = \rho\, (v_\infty \cos \theta + K) + \frac{1}{2} ai \neq 0 \text{ on } L$$

Hence according to the terminology of Muskhelishvili [2], equation (2.8) is a singular integral equation of a standard type.

The index of this equation is

$$\varkappa_* = \frac{1}{2\pi i}\left[\ln \frac{\rho(v_\infty \cos\theta + K) + {}^1/_2\, ia}{\rho(v_\infty \cos\theta + K) - {}^1/_2\, ia}\right]_L = 0$$

Thus equation (2.8), according to the same terminology, will be a Quasi-Fredholm equation and can be solved by the method of regularisation.

The regular type operator for equation (2.8) can be taken, for example, to be

$$M\Psi \equiv \rho(v_\infty \cos\nu + K)\Psi(t_0) - \frac{1}{\pi i}\int_L \frac{-{}^1/_2\, ia\Psi(t)}{t - t_0}\,dt$$

and after the application of this operator to both sides of equation (2.8), Fredholm's equation is obtained, which will be equivalent to (2.8). It is possible also to regularise equation (2.8) by using the method of Carleman and Vekua.

For the study of the flow of a potential stream without circulation round a uniformly permeable, smooth contour, the reasoning mentioned above remains valid, only it is necessary to take $K = 0$.

By way of example, we consider the problem of the flow of a potential stream round a uniformly permeable circle of radius R. In this case we take the centre of the circle to lie at the origin of the coordinate axes and ϕ and ϕ_0 to be the central angles of the points s and s_0. It follows immediately from the figure, therefore, that

$$\cos\nu = -\sin\varphi_0, \qquad \sin\nu = \cos\varphi_0, \qquad \lambda = \frac{\varphi - \varphi_0}{2}, \qquad r = 2R\sin\frac{\varphi - \varphi_0}{2}$$

$$ds = Rd\varphi, \quad \frac{\cos\lambda}{r}ds = \frac{1}{2}\operatorname{ctg}\frac{\varphi - \varphi_0}{2}\,d\varphi, \quad \frac{\sin\lambda}{r}ds = \frac{1}{2}\,d\varphi$$

Then the value of K, defined by the integral (2.4), will equal

$$\frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\lambda}{r}\,ds = \frac{1}{4\pi}\int_0^{2\pi}\gamma(\varphi)\,d\varphi = K_0$$

and, after a change of variable, the basic equation (1.17) (or, alternatively, equation (2.8)) may be written

$$\gamma(\varphi_0)\rho(v_\infty \sin\varphi_0 - K_0) + \frac{a}{4\pi}\int_0^{2\pi}\gamma(\varphi)\operatorname{ctg}\frac{\varphi - \varphi_0}{2}\,d\varphi = -av_\infty\cos\varphi_0 \qquad (2.9)$$

Applying to both sides of this singular equation the regular operator

$$M\Psi \equiv \rho(v_\infty \sin\varphi_0 - K_0)\Psi(\varphi_0) - \frac{{}^1/_2 a}{2\pi}\int_0^{2\pi}\Psi(\varphi)\operatorname{ctg}\frac{\varphi - \varphi_0}{2}\,d\varphi$$

we get a Fredholm equation, the unique solution of which will be

$$\gamma(\varphi_0) = -\frac{4a\rho v_\infty^{\,2}\sin\varphi_0\cos\varphi_0 + 2a^2(v_\infty\sin\varphi_0 - K_0)}{4\rho^2(v_\infty\sin\varphi_0 - K_0)^2 + a^2} \qquad (2.10)$$

This function gives the solution of the problem of the flow of a potential stream past a uniformly permeable circle.

As $a \to \infty$ in the limit, the equation (2.9) becomes

$$\frac{1}{4\pi}\int_0^{2\pi} \gamma(\varphi)\operatorname{ctg}\frac{\varphi-\varphi_0}{2}\,d\varphi = -v_\infty\cos\varphi_0 \tag{2.11}$$

the solution of which is

$$\gamma(\varphi_0) = -2v_\infty\sin\varphi_0 + C \tag{2.12}$$

a function giving the solution of the problem of the flow of a potential stream past a compact circle. Moreover, in the flow past a compact circle with circulation, it follows from the limiting value of the relation (2.10) that it is necessary to choose $C = 2K_0$ and for the flow without circulation to take $C = 0$.

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